

Classification of Casimirs in 2D hydrodynamics

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To the memory of Vladimir Igorevich Arnold

Abstract

We describe a complete list of Casimirs for 2D Euler hydrodynamics on a surface without boundary: we define generalized enstrophies which, along with circulations, form a complete set of invariants for coadjoint orbits of area-preserving diffeomorphisms on a surface. We also outline a possible extension of main notions to the boundary case and formulate several open questions in that setting.

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1 Introduction

The famous V. Arnold stability criterion gives a sufficient condition for stability of a steady two-dimensional flow: the flow is stable provided that the second variation of the energy restricted to the set of isovorticed fields is sign-definite. This criterion was generalized in many

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ways: to magnetohydrodynamics, to stratified fluids, to systems with additional symmetries, furthermore, such variations were studied in higher dimensions, perturbation methods were applied to show when instabilities arise, etc.

The knowledge of invariants of isovorticed fields becomes of utmost importance for applying this criterion or for developing its generalizations. The reason is that the criterion is based on defining a new functional as a combination of the fluid energy and those invariants. The latter are often called Casimirs of 2D flows, or enstrophies, or invariants of coadjoint orbits in two-dimensional hydrodynamics. More than once Arnold posed the problem of classification of isovorticed fields, as well as related problem of minimization of Dirichlet functional in 2D for initial functions of different topology.

While it has been known for a long time that enstrophies are first integrals of 2D incompressible fluid flows, a complete classification of generic Casimirs in 2D was obtained only recently in [6, 5]. Here we revisit and develop that classification by comparing it to other known classification of coadjoint orbits for diffeomorphism groups in one dimension. To describe the orbit classification we first present an axillary problem of classification of simple Morse functions with respect to area-preserving diffeomorphisms of a surface. It is convenient first to formulate the invariants as structures related to so-called Reeb graphs of functions.

Recall that the motion of an inviscid incompressible fluid filling an n -dimensional Riemannian manifold M is governed by the hydrodynamical *Euler equation*

$$\partial_t u + \nabla_u u = -\nabla p \quad (1)$$

on the divergence-free velocity field u of a fluid flow in M . Here $\nabla_u u$ stands for the Riemannian covariant derivative of the field u along itself, while the function p is determined by the divergence-free condition up to an additive constant.

In this paper we consider the case of a surface, $n = 2$. In this setting, the vorticity of the fluid can be regarded as the function $F = du^\flat/\omega$, where u^\flat is the 1-form metric-related to the vector field u on the surface, and ω is the Riemannian area form. (For Euclidean metric and $u = u_1\partial/\partial x_1 + u_2\partial/\partial x_2$ the vorticity function is $F = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$.) According to Kelvin's law, the vorticity function is "frozen into" the incompressible flow. This fact allows one to define Casimirs, i.e. first integrals of the Euler equation valid for any Riemannian metric. Namely, it is well known that enstrophies, i.e. all moments

$$m_i(F) := \int_M F^i \omega, \quad i = 0, 1, 2, \dots$$

of the vorticity function F , are Casimirs. These quantities are invariants of the natural action of the group of area-preserving diffeomorphisms of the surface M .

In the case of a flow in a two-sphere whose Morse vorticity function has one maximum and one minimum such enstrophy invariants form a complete set of Casimirs, see [5], cf. [4], while for more complicated functions and domains it is not so. Indeed, the set of all enstrophies is known to be incomplete for flows with generic vorticities: there are non-diffeomorphic vorticities with the same values of enstrophies, see Section 4.

In this paper we give a complete description of Casimir invariants for flows of an ideal 2D fluid with simple Morse vorticity functions. We define generalized enstrophies in terms of measured Reeb graphs and prove that they together with the set of circulations form a complete list of Casimirs in 2D hydrodynamics.

Example 1.1. The following example gives a glimpse of the basic constructions. The graph Γ_F , called the Reeb graph (also called Kronrod graph), is the set of connected components of

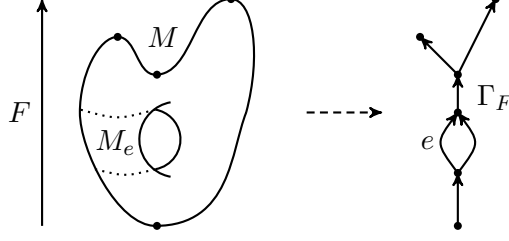


Figure 1: Reeb graph for a height function with two maxima on a torus.

the levels of a height function F on a surface M , see Figure 1. Critical points of F correspond to the vertices of the graph Γ_F . This graph comes with a natural parametrization by the values of F . For a symplectic surface M its area form ω induces a measure μ on the graph, which satisfies certain properties. This measured Reeb graph of the function F is a complete invariant of the function F with respect to the action of area-preserving diffeomorphisms of the surface:

Theorem A (= Theorem 3.8). *The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondence between simple Morse functions on M up to symplectomorphisms and measured Reeb graphs compatible with M .*

To obtain numerical invariants from this measured graph Γ_F one can consider for each edge $e \in \Gamma_F$ the preimage $M_e \subset M$ bounded by the corresponding critical levels of F . Then infinitely many moments

$$I_{i,e}(F) := \int_{M_e} F^i \omega, \quad i = 0, 1, 2, \dots$$

of the function F over each M_e (or, equivalently, the moments of the induced function on each edge the graph) are invariants of the $\text{SDiff}(M)$ -action, i.e., the action on the function F by symplectomorphisms of M .

The problem of classification of hydrodynamical Casimirs (i.e., invariants of coadjoint action) for Morse coadjoint orbits includes the above problem for invariant classification of a function, since all invariants of vorticity are Casimirs. These two problems coincide for a sphere, as the Reeb graph in that case is a tree and the vorticity function fully determines the coadjoint orbit. For surfaces of higher genus the Reeb graph has nontrivial first homology group, $\dim H_1(\Gamma_F) = \text{genus}(M) = \varkappa$ and to describe the orbit one needs also specify the circulations of the field around \varkappa cycles on the surface.

In order to classify coadjoint orbits of the symplectomorphism group we introduce a notion of an anti-derivative, or circulation function, for a Reeb graph. It turns out that such anti-derivatives form a finite-dimensional space of dimension equal to the first Betti number of the graph. Therefore the space of coadjoint orbits of the symplectomorphism group of a surface is a bundle over the space of fluid vorticities, where fiber coordinates can be thought of as circulations, see details in Section 3.3.

Theorem B (= Corollary 4.3). *A complete set of Casimirs for the 2D Euler equation in a neighborhood of a Morse-type coadjoint orbit is given by the moments $I_{i,e}$ for each edge $e \in \Gamma$, $i = 0, 1, 2, \dots$, and all circulations of the velocity v over cycles in the singular levels of the vorticity function F on M .*

Remark 1.2. It is interesting to compare the description of $\text{SDiff}(M)$ -orbits for a surface M with the classification of coadjoint orbits of the group $\text{Diff}(S^1)$ of circle diffeomorphisms [7].

Its Lie algebra is $\mathfrak{vect}(S^1)$ and the (smooth) dual space $\mathfrak{vect}^*(S^1)$ is identified with the space of quadratic differentials on the circle, $QD(S^1) := \{F(x)(dx)^2 \mid F \in C^\infty(S^1, \mathbb{R})\}$. For a generic function F changing sign on the circle, a complete set of invariants is given by the “weights”

$$I_{a_k}(F) := \int_{a_k}^{a_{k+1}} \sqrt{|F(x)|} dx$$

of the quadratic differential between every two consecutive zeros $a_k < a_{k+1}$ of $F(x)$ on the circle S^1 . These orbits are of finite codimension equal to the number of zeros. In a family of functions, where two new zeros, say a'_k and a''_k , appear between original zeros a_k and a_{k+1} : $a_k < a'_k < a''_k < a_{k+1}$, one gains two extra Casimir functions, $I_{a'_k}$ and $I_{a''_k}$, and hence the codimension of the orbit jumps up by 2.

Similarly, for functions or coadjoint orbits of symplectomorphisms on a 2D surface, the appearance of a new pair of critical points, say, a saddle and a local maximum for a function, leads to splitting of one edge in two and, in addition to that, to the appearance of a new edge in the corresponding Reeb graph, and hence to two new families of Casimirs related to those extra edges, as in Example 1.1.

Note that for the action of the subgroup consisting of symplectomorphisms in the connected component of the identity, one encounters additional discrete invariants related to pants decompositions and possible projections of the surface to the graph. For the group of Hamiltonian diffeomorphisms the above set of orbit invariants is supplemented by fluxes of diffeomorphisms across certain cycles on the surface M , see details in [6].

Motivation for this type of classification problems is coming from fluid dynamics. For instance, steady fluid flows are conditional extrema of the energy functional on the sets of isovorticed fields, so Casimirs allow one to single out such sets in order to introduce appropriate Lagrange multipliers. Furthermore, Casimirs in fluid dynamics are a cornerstone of the energy-Casimir method for the study of hydrodynamical stability, see e.g. [2].

In addition to hydrodynamics, the results can be also used for the extension of the orbit method to infinite-dimensional groups of 2D diffeomorphisms. According to this method, adjacency of coadjoint orbits of a group or its central extension mimics families of appropriate representations of the corresponding group. This method turned out to be effective for affine groups and the Virasoro-Bott group, so one may hope to apply it to 2D diffeomorphisms and current groups as well. Finally, note that all objects in the present paper are infinitely smooth (see the case of finite smoothness in [6]). To the best of our knowledge, a complete description of Casimirs in 2D fluid dynamics has not previously appeared in the literature in a self-contained form, while various partial results could be found in [3, 9, 10, 11, 5]. In the last section we present a few examples, show how the main notions can be extended to the case of surfaces with boundary, emphasize the main difficulties and formulate open questions in the latter setting.

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2 The hydrodynamical Euler equation

2.1 Geodesic and Hamiltonian frameworks of the Euler equation

Consider an inviscid incompressible fluid filling a closed (i.e., compact and without boundary) n -dimensional Riemannian manifold M with the Riemannian volume form μ . Arnold [2] showed that the Euler equation can be regarded as an equation of the geodesic flow on the group $\text{SDiff}(M) := \{\phi \in \text{Diff}(M) \mid \phi^*\mu = \mu\}$ of volume-preserving diffeomorphisms of M with respect to a right-invariant metric on the group given at the identity by the L^2 -norm of the fluid's velocity field. This geodesic description implies the following Hamiltonian framework for the Euler equation. Consider the (smooth) dual space $\mathfrak{g}^* = \mathfrak{svect}^*(M)$ to the space $\mathfrak{g} = \mathfrak{svect}(M) = \{u \in \mathfrak{vect}(M) \mid L_u\mu = 0\}$ of divergence-free vector fields on M . This dual space has a natural description as the space of cosets $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$, where $\Omega^k(M)$ is the space of smooth k -forms on M . For a 1-form α on M its coset of 1-forms is

$$[\alpha] = \{\alpha + df \mid \text{for all } f \in C^\infty(M)\} \in \Omega^1(M)/d\Omega^0(M).$$

The pairing between cosets and divergence-free vector fields is given by $\langle [\alpha], u \rangle := \int_M \alpha(u) \omega$ for any field $u \in \mathfrak{svect}(M)$. (This pairing is well-defined on cosets because the latter integral vanishes for any exact 1-form α and any $u \in \mathfrak{svect}(M)$.) The coadjoint action of the group $\text{SDiff}(M)$ on the dual \mathfrak{g}^* is given by the change of coordinates in (cosets of) 1-forms on M by means of volume-preserving diffeomorphisms.

The Riemannian metric (\cdot, \cdot) on the manifold M allows one to identify the Lie algebra and its (smooth) dual by means of the so-called inertia operator: given a vector field u on M one defines the 1-form $\alpha = u^\flat$ as the pointwise inner product with the velocity field u : $u^\flat(v) := (u, v)$ for all $v \in T_x M$. Note also that divergence-free fields u correspond to co-closed 1-forms u^\flat . The Euler equation (1) rewritten on 1-forms $\alpha = u^\flat$ is $\partial_t \alpha + L_u \alpha = -dP$ for an appropriate function P on M . In terms of the cosets of 1-forms $[\alpha]$, the Euler equation on the dual space \mathfrak{g}^* takes the form

$$\partial_t [\alpha] + L_u [\alpha] = 0. \quad (2)$$

The Euler equation (2) on $\mathfrak{g}^* = \mathfrak{svect}^*(M)$ turns out to be a Hamiltonian equation with the Hamiltonian functional \mathcal{H} given by the fluid's kinetic energy, $\mathcal{H}([\alpha]) = \frac{1}{2} \int_M (u, u) \mu$ for $\alpha = u^\flat$. The corresponding Poisson structure is given by the natural linear Lie-Poisson bracket on the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} , see details in [2, 3]. The corresponding Hamiltonian operator is given by the Lie algebra coadjoint action ad_u^* , which in the case of the diffeomorphism group corresponds to the Lie derivative: $\text{ad}_u^* = L_u$. Its symplectic leaves are coadjoint orbits of the corresponding group $\text{SDiff}(M)$. All invariants of the coadjoint action, also called Casimirs, are first integrals of the Euler equation for *any choice* of Riemannian metric. The main result of this paper is a complete characterization of Casimirs for the 2D Euler equation on closed surfaces, see Section 4.

2.2 Vorticity and Casimirs of the 2D Euler equation

Recall that according to the Euler equation (2) the coset of 1-forms $[\alpha]$ evolves by a volume-preserving change of coordinates, i.e. during the Euler evolution it remains in the same coadjoint orbit in \mathfrak{g}^* . Introduce the *vorticity 2-form* $\xi := du^\flat$ as the differential of the 1-form $\alpha = u^\flat$ and note that the vorticity exact 2-form is well-defined for cosets $[\alpha]$: 1-forms α in

the same coset have equal vorticities $\xi = d\alpha$. The corresponding Euler equation assumes the vorticity (or Helmholtz) form

$$\partial_t \xi + L_u \xi = 0,$$

which means that the vorticity form is transported by (or “frozen into”) the fluid flow (Kelvin’s theorem). The definition of vorticity ξ as an exact 2-form $\xi = du^\flat$ makes sense for a manifold M of any dimension. In the case of two-dimensional oriented surfaces M the group $\text{SDiff}(M)$ of volume-preserving diffeomorphisms of M coincides with the group $\text{Symp}(M)$ of symplectomorphisms of M with the area form $\mu = \omega$ given by the symplectic structure. In what follows, in 2D our main object of consideration is the vorticity function F related to the vorticity 2-form $\xi = F\omega$ by means of the symplectic structure.

Remark 2.1. The fact that the vorticity 2-form ξ is “frozen into” the incompressible flow allows one to define first integrals of the hydrodynamical Euler equation valid for any Riemannian metric on M . In 2D the Euler equation on M is known to possess infinitely many so-called *enstrophy invariants* $m_\lambda(F) := \int_M \lambda(F) \omega$, where $\lambda(F)$ is an arbitrary function of the vorticity function F . In particular, the enstrophy moments $m_i(F) := \int_M F^i \omega$ are conserved quantities for any $i \in \mathbb{Z}_{\geq 0}$. These Casimir invariants are fundamental in the study of nonlinear stability of 2D flows, and in particular, were the basis for Arnold’s stability criterion in ideal hydrodynamics, see [2, 3]. In the energy-Casimir method one studies the second variation of the energy functional with an appropriately chosen combination of Casimirs.

In the case of a flow in an annulus with a vorticity function without critical points such invariants, along with the circulation along one of the two boundary components, form a complete set of Casimirs [4], while for more complicated functions and domains it is not so, see Section 4. In Section 4 we give a complete description of Casimirs in the general setting of Morse vorticity functions on two-dimensional surfaces.

3 Coadjoint orbits of the symplectomorphism group

Before classifying coadjoint orbits of the symplectomorphism group we solve the problem of finding a complete invariant for a function on a closed symplectic surface. (See Section 5.2 for the case with boundary.)

3.1 Simple Morse functions and measured Reeb graphs

Definition 3.1. Let M be a closed connected surface. A Morse function $F: M \rightarrow \mathbb{R}$ is called *simple* if any F -level contains at most one critical point. (Here and below under F -level we mean a connected component of the set $F = \text{const.}$)

With each simple Morse function $F: M \rightarrow \mathbb{R}$, one can associate a graph. This graph Γ_F is defined as the space of F -levels with the induced quotient topology. Each vertex of this graph corresponds to a critical level of the function F . The function F on M descends to a function f on the graph Γ_F . It is also convenient to assume that Γ_F is oriented: edges are oriented in the direction of increasing f .

Example 3.2. Figure 1 shows level curves of a simple Morse function on a torus and the corresponding graph Γ_F .

Definition 3.3. A *Reeb graph* (Γ, f) is an oriented connected finite graph Γ with a continuous function $f: \Gamma \rightarrow \mathbb{R}$ which satisfy the following properties.

- i) All vertices of Γ are either 1-valent or 3-valent.
- ii) For each 3-valent vertex, there are either two incoming and one outgoing edge, or vice versa.
- iii) The function f is strictly monotonous on each edge of Γ , and the edges of Γ are oriented towards the direction of increasing f .

It is a standard result from Morse theory that the graph Γ_F associated with a simple Morse function $F: M \rightarrow \mathbb{R}$ on an orientable connected surface M is a Reeb graph in the sense of Definition 3.3. We will call this graph the Reeb graph of the function F . Note that Reeb graphs classify simple Morse functions on M up to diffeomorphisms.

In what follows, we assume that the surface M is endowed with an area (i.e., symplectic) form ω . We are interested in the classification problem for simple Morse functions up to area-preserving (i.e., symplectic) diffeomorphisms. It turns out that this classification can be given in terms of so-called *log-smooth measures* on Reeb graphs.

Definition 3.4. Let Γ be a Reeb graph. Assume that e_0, e_1 , and e_2 are three edges of Γ which meet at a 3-valent vertex v . Then e_0 is called the *trunk* of v , and e_1, e_2 are called *branches* of v if either e_0 is an outgoing edge for v , and e_1, e_2 are its incoming edges, or vice versa.

Definition 3.5. A measure μ on a Reeb graph (Γ, f) is called *log-smooth* if it has the following properties:

- i) It has a C^∞ -smooth non-vanishing density $d\mu/df$ at interior points and 1-valent vertices of Γ .
- ii) At 3-valent vertices, the measure μ has logarithmic singularities. More precisely, consider a 3-valent vertex v of Γ . Without loss of generality assume that $f(v) = 0$ (if not, we replace f by $\tilde{f}(x) := f(x) - f(v)$). Let e_0 be the trunk of v , and let e_1, e_2 be the branches of v . Then there exist functions $\psi, \eta_0, \eta_1, \eta_2$ of one variable, C^∞ -smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and such that for any point $x \in e_i$ sufficiently close to v , we have

$$\mu([v, x]) = \varepsilon_i \psi(f(x)) \ln |f(x)| + \eta_i(f(x)),$$

where $\varepsilon_0 = 2$, $\varepsilon_1 = \varepsilon_2 = -1$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 = 0$.

Definition 3.6. A Reeb graph (Γ, f) endowed with a log-smooth measure μ is called a *measured Reeb graph*.

If a surface M is endowed with an area form ω , then the Reeb graph Γ_F of any simple Morse function $F: M \rightarrow \mathbb{R}$ has a natural structure of a measured Reeb graph. The measure μ on Γ_F is defined as the pushforward of the area form on M under the natural projection $\pi: M \rightarrow \Gamma_F$.

It turns out that there is a one-to-one correspondence between simple Morse functions on M , considered up to symplectomorphisms, and measured Reeb graphs satisfying the following natural compatibility conditions:

Definition 3.7. Let M be a connected closed surface endowed with a symplectic form ω . A measured Reeb graph (Γ, f, μ) is *compatible with* (M, ω) if (i) the dimension of $H_1(\Gamma, \mathbb{R})$ is equal to the genus of M and (ii) the volume of Γ with respect to the measure μ is equal to the volume of M : $\int_\Gamma d\mu = \int_M \omega$.

Theorem 3.8. [6] *The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondence between simple Morse functions on M up to a symplectomorphism and measured Reeb graphs compatible with M .*

3.2 Antiderivatives on graphs

In order to pass from the above classification of simple Morse functions on symplectic surfaces to the classification of coadjoint orbits of the group $\text{SDiff}(M)$, we need to introduce the notion of the antiderivative of a density on a graph. Let Γ be an oriented graph. Let also ρ be a *density* on Γ , i.e. a finite signed Borel measure.

Definition 3.9. A function $\lambda: \Gamma \setminus V \rightarrow \mathbb{R}$ defined and continuous on the graph Γ outside its set of vertices $V = V(\Gamma)$ is called an *antiderivative of the density ρ* if it has the following properties.

- i) It has at worst jump discontinuities at vertices, which means that for any vertex $v \in V$ and any edge $e \ni v$, there exists a finite limit $\lim_{x \xrightarrow{e} v} \lambda(x)$, where $x \xrightarrow{e} v$ means “as x tends to v along the edge e ”.
- ii) Assume that x, y are two interior points of some edge $e \in \Gamma$, and that e is pointing from x towards y . Then λ satisfies the Newton-Leibniz formula

$$\lambda(y) - \lambda(x) = \rho([x, y]).$$

- iii) For a vertex v of Γ the function λ satisfies the Kirchhoff rule at v :

$$\sum_{e \rightarrow v} \lim_{x \rightarrow v} \lambda(x) = \sum_{e \leftarrow v} \lim_{x \rightarrow v} \lambda(x),$$

where the notation $e \rightarrow v$ stands for the set of edges pointing at the vertex v , and $e \leftarrow v$ stands for the set of edges pointing away from v .

Proposition 3.10. *For an oriented graph Γ a density ρ on Γ admits an antiderivative if and only if $\rho(\Gamma) = 0$. Furthermore, if a density ρ on Γ admits an antiderivative, then the set of antiderivatives of ρ is an affine space whose associated vector space is the homology group $H_1(\Gamma, \mathbb{R})$.*

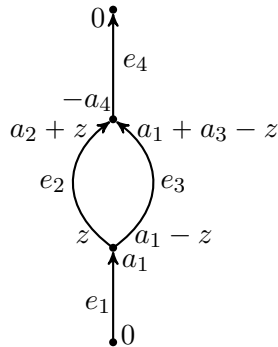


Figure 2: The space of antiderivatives on a graph of genus one.

Example 3.11. Consider the graph Γ depicted in Figure 2. Let ρ be a density on Γ such that $\rho(e_i) = a_i$, where the numbers a_i satisfy $a_1 + a_2 + a_3 + a_4 = 0$ (so that the density ρ admits an antiderivative). The numbers near vertices in the figure stand for the limits of the antiderivative λ of ρ . The space of such antiderivatives has one parameter z (by the proposition above the space of antiderivatives is one-dimensional).

3.3 Classification of coadjoint orbits

Let M be a closed connected surface endowed with a symplectic form ω . Recall that the regular dual $\mathfrak{vect}^*(M)$ of the Lie algebra $\mathfrak{vect}(M)$ of divergence-free vector fields on a surface M is identified with the space $\Omega^1(M)/d\Omega^0(M)$ of smooth 1-forms modulo exact 1-forms on M . The coadjoint action of a $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$ is given by the change of coordinates in (cosets of) 1-forms on M by means of a symplectic diffeomorphism: $\text{Ad}_\Phi^*[\alpha] = [\Phi^*\alpha]$. To describe orbits of the coadjoint action of $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$, consider the mapping $\text{curl}: \Omega^1(M)/d\Omega^0(M) \rightarrow C^\infty(M)$ given by taking the vorticity function

$$\text{curl}[\alpha] := \frac{d\alpha}{\omega}.$$

(One can view this map as taking the vorticity of a vector field $u = \alpha^\sharp$.) Note that the image of the mapping curl is the space of functions with zero mean.

By definition, the mapping curl is equivariant with respect to the $\text{SDiff}(M)$ action: if cosets $[\alpha], [\beta] \in \mathfrak{vect}^*(M)$ belong to the same coadjoint orbit, then the functions $\text{curl}[\alpha]$ and $\text{curl}[\beta]$ are related by a symplectic diffeomorphism. In particular, if $\text{curl}[\alpha]$ is a simple Morse function, then so is $\text{curl}[\beta]$.

Definition 3.12. We say that a coset of 1-forms $[\alpha] \in \mathfrak{vect}^*(M)$ is *Morse-type* if $\text{curl}[\alpha]$ is a simple Morse function. A coadjoint orbit $\mathcal{O} \subset \mathfrak{vect}^*(M)$ is *Morse-type* if any coset $[\alpha] \in \mathcal{O}$ is Morse-type (equivalently, if at least one coset $[\alpha] \in \mathcal{O}$ is Morse-type).

Let $[\alpha] \in \mathfrak{vect}^*(M)$ be Morse-type, and let $F := \text{curl}[\alpha]$. Consider the measured Reeb graph Γ_F . Since curl is an equivariant mapping, this graph is invariant under the coadjoint action of $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$. However, this invariant is not complete if M is not simply connected (i.e., if M is not a sphere S^2). To construct a complete invariant, we endow the graph Γ_F with a *circulation function* constructed as follows. Let $\pi: M \rightarrow \Gamma_F$ be the natural projection. Take any point x lying in the interior of some edge $e \in \Gamma_F$. Then $\pi^{-1}(x)$ is a circle. It is naturally oriented as the boundary of the set of smaller values of F . The integral of α over $\pi^{-1}(x)$ does not depend on the choice of a representative $\alpha \in [\alpha]$. Thus, we obtain a function $\mathfrak{c}: \Gamma_F \setminus V(\Gamma_F) \rightarrow \mathbb{R}$ given by

$$\mathfrak{c}(x) := \int_{\pi^{-1}(x)} \alpha. \quad (3)$$

Note that in the presence of a metric on M , the value $\mathfrak{c}(x)$ is the circulation over the level $\pi^{-1}(x)$ of the vector field α^\sharp dual to the 1-form α .

Proposition 3.13. *For any Morse-type coset $[\alpha] \in \mathfrak{vect}^*(M)$, the function \mathfrak{c} given by formula (3) is an antiderivative of the density $\rho(I) := \int_I f d\mu$ in the sense of Definition 3.9.*

Remark 3.14. This density ρ is the pushforward of the vorticity 2-form $d[\alpha]$ from the surface to the Reeb graph.

Proof of Proposition 3.13. The proof is straightforward and follows from the Stokes formula and additivity of the circulation integral. \square

Definition 3.15. Let (Γ, f, μ) be a measured Reeb graph. A *circulation function* \mathbf{c} on Γ is an antiderivative of the density $\rho(I) := \int_I f d\mu$. A measured Reeb graph endowed with a circulation function is called a *circulation graph*.

So, with any Morse-type coset $[\alpha] \in \mathbf{sVect}^*(M)$ we associate a circulation graph $\Gamma_{[\alpha]}$.

Theorem 3.16. [6] *Let M be a compact connected symplectic surface. Then Morse-type coadjoint orbits of $\mathrm{SDiff}(M)$ are in one-to-one correspondence with circulation graphs $(\Gamma, f, \mu, \mathbf{c})$ compatible with M . In other words, the following statements hold:*

- i) For a symplectic surface M Morse-type cosets $[\alpha], [\beta] \in \mathbf{sVect}^*(M)$ lie in the same orbit of the $\mathrm{SDiff}(M)$ coadjoint action if and only if circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to these cosets are isomorphic.*
- ii) For each circulation graph Γ which is compatible¹ with M , there exists a Morse-type $[\alpha] \in \mathbf{sVect}^*(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, f, \mu, \mathbf{c})$.*

Remark 3.17. The space of circulation graphs for a given vorticity function has dimension equal to $\dim H_1(\Gamma, \mathbb{R}) = \text{genus}(M)$. Given a vorticity function, in order to define a circulation graph uniquely it suffices to consider a measured graph and set the values of \varkappa circulations, where $\varkappa = \dim H_1(\Gamma, \mathbb{R})$, one value on each cycle of Γ . In other words, one can consider \varkappa points p_i , $i = 1, \dots, \varkappa$ on the graph Γ so that cuts at those points turn $\Gamma \setminus \{p_1, \dots, p_\varkappa\}$ into a tree, i.e. graph without cycles. Then by prescribing values of an antiderivative at all those points p_i we determine the circulation function on the graph uniquely.

Remark 3.18. For fluid dynamics we are interested only in connected components of coadjoint orbits, i.e. orbits with respect to the group $\mathrm{SDiff}_0(M)$, which is the connected component of the identity in the group $\mathrm{SDiff}(M)$ of all symplectomorphisms of M . To classify orbit invariants for the connected group $\mathrm{SDiff}_0(M)$ one needs to supplement invariants for $\mathrm{SDiff}(M)$ given by Theorem 3.16 by adding certain discrete invariants related to pants decompositions of the surface M and Dehn half-twists. A complete list of the corresponding invariants is given by Theorem 4.7 in [6], which we refer to for more detail. Note that those discrete invariants do not affect the list of Casimirs we are interested in here.

4 Casimir invariants of the 2D Euler equation

Above we classified coadjoint orbits of the group $\mathrm{SDiff}(M)$ in terms of graphs with certain additional structures, see Theorem 3.16. However, for applications, it is important to describe numerical invariants of the coadjoint action, i.e., Casimir functions. We begin with the description of such invariants for functions on symplectic surfaces.

Let (M, ω) be a closed connected symplectic surface, and let F be a simple Morse function on M . With each edge e of the measured Reeb graph $\Gamma_F = (\Gamma, f, \mu)$, one can associate an infinite sequence of moments

$$m_{i,e}(F) = \int_e f^i d\mu = \int_{M_e} F^i \omega,$$

¹See Definition 3.7 for compatibility of a graph and a surface.

where $i = 0, 1, 2, \dots$, and $M_e = \pi^{-1}(e)$ for the natural projection $\pi: M \rightarrow \Gamma$. Obviously, the moments $m_{i,e}(F)$ are invariant under the action of $\text{SDiff}(M)$ on simple Morse functions. Moreover, they form a complete set of invariants in the following sense:

Theorem 4.1. *Let (M, ω) be a closed connected symplectic surface, and let F and G be simple Morse functions on M . Assume that $\phi: \Gamma_F \rightarrow \Gamma_G$ is an isomorphism of abstract directed graphs which preserves moments on all edges. Then Γ_F and Γ_G are isomorphic as measured Reeb graphs, and there exists a symplectomorphism $\Phi: M \rightarrow M$ such that $\Phi_*F = G$.*

Proof. Consider an edge $e = [v, w] \in \Gamma_F$. Pushing forward the measure μ on e by means of the homeomorphism $f: e \rightarrow [f(v), f(w)] \subset \mathbb{R}$, we obtain a measure μ_f on the interval $I_f = [f(v), f(w)]$, whose moments coincide with the moments of μ at e . Repeating the same construction for the measure on Γ_G , we obtain another measure μ_g , which is defined on the interval $I_g = [g(\phi(v)), g(\phi(w))]$ and has the same moments as μ_f .

Now, consider any closed interval $I \subset \mathbb{R}$ which contains both I_f and I_g . Then the measures μ_f, μ_g may be viewed as measures on the interval I supported at I_f and I_g respectively. The moments of the measures μ_f, μ_g on I coincide, so, by the uniqueness theorem for the Hausdorff moment problem (see Remark 4.2), we have $\mu_f = \mu_g$, which implies the proposition. \square

Remark 4.2. The Hausdorff moment problem gives the following necessary and sufficient condition: a sequence of numbers m_k can be the set of moments $m_k(\lambda) = \int_0^1 \lambda^k d\mu(\lambda)$ of some Borel measure μ supported on the interval $[0, 1]$ if and only if it satisfies the so-called monotonicity conditions. The latter are linear inequalities on m_k , which can be derived from the relations $\int_0^1 \lambda^k (1 - \lambda)^n d\mu(\lambda) \geq 0$ for all integer $k, n \geq 0$, where the left-hand side is expressed in terms of m_k . (For instance, $m_3 - 2m_4 + m_5 = \int_0^1 \lambda^3 (1 - \lambda)^2 d\mu(\lambda) \geq 0$.) In our case, replacing λ by the parameter f we only employ the statement that the measure $\mu(f)$ is fully determined by the set $\{m_k, k = 0, 1, 2, \dots\}$.

In fact, it turns out that under certain regularity conditions the measure μ can be found in a constructive way from the moment sequence $\{m_k\}$. Assume, e.g. that $\mu(\lambda)$ is supported on a segment $[-L, L]$ and is given by a smooth positive density function $d\mu(\lambda) = w(\lambda) d\lambda$. Then consider the function Φ of a complex variable λ defined by

$$\Phi(\lambda) = \int_{[-L, L]} \frac{d\mu(z)}{\lambda - z} = \sum_{k \geq 0} \frac{m_k}{\lambda^{k+1}}.$$

The integral expression shows that Φ is defined and holomorphic in the complement of the real segment $[-L, L] \subset \mathbb{R}$. (One can also show that $|m_k| \leq CL^k$ and hence the series converges for $|\lambda| > L$.) Now the measure density $w(\lambda)$ can be recovered from Φ as its normalized jump across the cut $[-L, L]$ in the real axis (see [1, 8]):

$$w(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} (\Phi(\lambda - i\epsilon) - \Phi(\lambda + i\epsilon)).$$

The above Theorem 4.1 allows one to describe Casimirs of the 2D Euler equation on M , i.e. invariants of the coadjoint action of the symplectomorphism group $\text{SDiff}(M)$. Let F be a Morse vorticity function of an ideal flow with velocity v on a closed surface M , and let Γ be its Reeb graph. The corresponding moments $m_{i,e}(F)$ for this vorticity are natural to call *generalized enstrophies*. Then group coadjoint orbits in the vicinity of an orbit with the vorticity function F are singled out as follows.

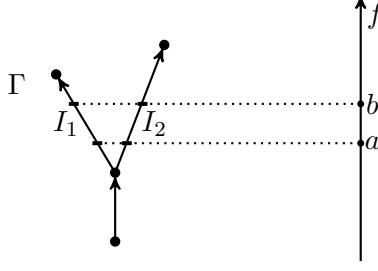


Figure 3: Modifying the measure on the edges.

Corollary 4.3. *A complete set of Casimirs of a 2D Euler equation in a neighborhood of a Morse-type coadjoint orbit is given by the moments $m_{i,e}$ for each edge $e \in \Gamma$, $i = 0, 1, 2, \dots$, and all circulations of the velocity v over cycles in the singular levels of F on M .*

Note that the (finite) set of required circulations can be sharpened by considering fewer quantities needed to describe the circulation function, as in Section 3.3.

Remark 4.4. As invariants of the coadjoint action of $\text{SDiff}(M)$, one usually considers total moments

$$m_i(F) = \int_M F^i \omega = \int_\Gamma f^i d\mu,$$

where $F = \text{curl}[\alpha]$ is the vorticity function, and (Γ, f, μ) is the measured Reeb graph of F . However, the latter moments do not form a complete set of invariants even in the case of a sphere or a disk.

Consider, for example, the measured Reeb graph (Γ, f, μ) depicted in Figure 3. Let μ' be any smooth measure on \mathbb{R} supported in $[a, b]$. Define a new measure $\tilde{\mu}$ on Γ by “moving some density from one branch to another”, i.e. by setting

$$\tilde{\mu} := \begin{cases} \mu + f^*(\mu') & \text{in } I_1, \\ \mu - f^*(\mu') & \text{in } I_2, \end{cases}$$

and $\tilde{\mu} := \mu$ elsewhere. Then $(\Gamma, f, \tilde{\mu})$ is again a measured Reeb graph. Moreover, for all total moments we have

$$\int_\Gamma f^k d\mu = \int_\Gamma f^k d\tilde{\mu}.$$

However, the measured graphs (Γ, f, μ) and $(\Gamma, f, \tilde{\mu})$ are not isomorphic and thus correspond to two different coadjoint orbits of $\text{SDiff}(S^2)$.

5 Examples and open questions

In this section we consider several examples and open questions related to the description of Casimirs for the case of a surface with boundary.

5.1 Generalized enstrophies and circulations

In the next example we look at a genus one surface M with an area form ω and a Morse height function F on it as in Figure 1. Consider the domain $M_e \subset M$ associated with each

edge e of the Reeb graph Γ_F . This domain $M_e = \pi^{-1}(e)$ is the preimage of the edge bounded by the corresponding critical levels of F . In this example there are 6 edges in the graph Γ_F . Consider the infinite set of all generalized enstrophies, i.e. all moments $\int_{M_e} F^k \omega$ of the vorticity function in these regions. All generalized enstrophies are Casimirs of the coadjoint action of the group $\text{SDiff}(M)$.

They do not exhaust all Casimirs, but in general must be supplemented by several circulations. In this example of a torus one needs to fix the value of one circulation (and, more generally, one value for each handle of the surface). For instance, one can fix a value of the circulation function at the lower boundary of domain M_e , corresponding to the bottom of edge e in Figure 1. (In Corollary 4.3 we mentioned circulations over all critical levels of F , which contains the one above, and several other circulations which are dependent on it.) Note that the set of all generalized enstrophies and circulations described in the corollary is not a “minimal set” of Casimirs, as the Hausdorff moment problem does not claim the minimality.

Remark 5.1. Recall that the dual $\mathfrak{g}^* = \mathfrak{svect}^*(M)$ to the Lie algebra $\mathfrak{g} = \mathfrak{svect}(M)$ of divergence-free vector fields on M consists of cosets of 1-forms $[\alpha] = \{\alpha + df \mid \text{for all } f \in C^\infty(M)\}$, elements of the quotient $\Omega^1(M)/d\Omega^0(M) = \mathfrak{svect}^*(M)$. The function $F = d\tilde{\alpha}/\omega$, as well as values $\mathfrak{c}(x) := \int_{\ell_x} \tilde{\alpha}$, are defined for any $[\tilde{\alpha}] \in \mathfrak{g}^* = \mathfrak{svect}^*(M)$ in a neighborhood of the coadjoint orbit of $[\alpha]$. Their invariant definition, i.e. the definition relying only on the choice of an area form, but not a metric on M , in a sense, explains the Casimir property of those quantities. On the other hand, the *interpretation* of those values as the set of *vector fields with given vorticity and circulations* (rather than the set of 1-forms), which are metric-related to the 1-forms, requires the presence of metric, and hence such a set is not $\text{SDiff}(M)$ -invariant.

5.2 The boundary case: Morse functions

Here we briefly describe the necessary changes in the classification theorems in the case of a surface M with boundary ∂M and main difficulties which arise in this setting. Now the group of area-preserving diffeomorphisms of a connected surface M has the Lie algebra consisting of divergence-free vector fields on M tangent to ∂M . As before, we first try to classify generic functions on M with respect to this group action, and then move to coadjoint orbits.

Definition 5.2. Let M be a compact connected surface with a possibly non-empty boundary ∂M . A Morse function $F: M \rightarrow \mathbb{R}$ is called *simple* if it satisfies the following conditions:

- i) F does not have critical points at the boundary;
- ii) the restriction of F to the boundary ∂M is a Morse function;
- iii) all critical values of F and its restriction $F|_{\partial M}$ are distinct².

Associate the following *Reeb graph* Γ_F defined as the space of F -levels to such a simple Morse function $F: M \rightarrow \mathbb{R}$. Each vertex of this graph corresponds either to a critical level of the function F or to a critical point of its restriction $F|_{\partial M}$ to the boundary. As before, the function F on M descends to a function f on the graph Γ_F , which allows us to orient the edges of Γ_F by increasing f .

Note that now noncritical levels of F are either circles or segments. We denote the corresponding edges of the Reeb graph by *solid lines* if they correspond to *circle levels* and by

²Slightly more generally, in agreement with Definition 3.1, one can assume that any F -level contains either at most one critical point or at most one critical point of the restriction (but not both).

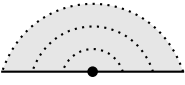

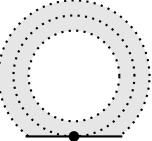

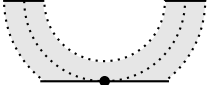

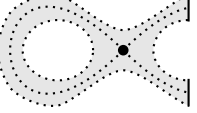

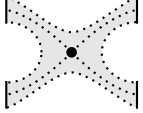

Type	Level Sets	Reeb Graph	Type	Level Sets	Reeb Graph
I			II		
III			IV		
V					

Figure 4: Five types of critical points on surfaces with boundary and type of vertices of the corresponding Reeb graphs. Solid lines correspond to pieces of the boundary, while dotted lines are connected components of level sets of the function.

dashed lines if they correspond to *segment levels*. In the boundary case, in addition to two types of vertices for solid lines, max/min corresponding to 1-valent vertices, and saddles corresponding to Y -type 3-valent vertices, we have 5 more types of vertices involving dashed lines.

Namely, as depicted on Figure 4 by employing this correspondence of solid and dashed edges of a Reeb graph to circular levels and segments respectively, one can have

- i) a min/max on the boundary, corresponding to a 1-valent vertex with a dashed line in the Reeb graph,
- ii) a min/max on the boundary, corresponding to a 2-valent vertex with one solid and one dashed line in the Reeb graph,
- iii) a min/max on the boundary, corresponding to a Y -type 3-valent vertex with three dashed lines in the Reeb graph,
- iv) a saddle point on M , corresponding to a Y -type 3-valent vertex with two dashed lines and a solid line coming together in the Reeb graph,
- v) a saddle point on M , see Figure 4, corresponding a 4-valent vertex of X -type between dashed edges in Γ_F .

Definition 5.3. A *Reeb graph* (Γ, f) is an oriented connected graph Γ with dashed and solid edges and a continuous function $f: \Gamma \rightarrow \mathbb{R}$ which satisfy the properties following from the above description of vertices.

As before, a simple Morse function $F: M \rightarrow \mathbb{R}$ on an orientable connected surface M with boundary can be associated with a Reeb graph Γ_F in the sense of Definition 3.3.

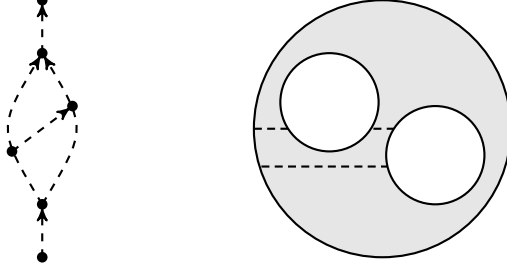


Figure 5: Dashed Reeb graph with $\dim H_1(\Gamma) = 2$ corresponding to both a disc with two holes and torus with one hole. Cutting the disk drawn here along the three dashed levels and then restoring the gluings with opposite orientations, one obtains a torus with one hole.

Example 5.4. It turns out one cannot reconstruct topology of the surface with boundary from its Reeb graph alone. For example, consider a dashed graph Γ with $\dim H_1(\Gamma) = 2$, see Figure 5. This graph corresponds to a disk with two holes, with the corresponding function given by the vertical coordinate y . Cutting the disk along the dashed level sets of y and then restoring the three gluings with opposite orientations, one obtains a torus with one hole. (Indeed, after the new gluings one obtains an oriented surface with the same Euler characteristic -1 , but with only one boundary component, hence a torus with a hole.) Since we cut the surface along level sets, the torus is naturally equipped with a simple Morse function whose graph coincides with the initial one.

Problem 5.5. *Describe the full information required from an abstract Reeb graph to reconstruct the corresponding surface with boundary.*

Further, we are interested in functions on surfaces equipped with an area form. As before, the Reeb graph Γ_F of any simple Morse function $F: M \rightarrow \mathbb{R}$ on a surface with boundary has a natural structure of a measured Reeb graph, where the measure μ on Γ_F is the pushforward of the area form on M . In the presence of an area form ω on M the Reeb graph gets endowed with an appropriately defined *log-smooth measure*.

Problem 5.6. *Describe the properties of the log-smooth measure μ on a Reeb graph (Γ, f) with solid and dashed edges. Namely, describe the asymptotics similar to Definition 3.5 at vertices of types (i)-(v).*

Solution of this problem will allow one to define an abstract measured Reeb graph as a Reeb graph (Γ, f) with solid and dashed edges and endowed with a log-smooth measure μ . The correspondence between abstract measured Reeb graphs and those corresponding to functions on surfaces with boundary now should include compatibility conditions for the surface and the Reeb graph beyond equality of total volumes $\int_{\Gamma} d\mu = \int_M \omega$ and of the corresponding homology, as it should account for certain discrete information discussed above.

Problem 5.7. *Describe the compatibility conditions for surfaces with boundary and area form and augmented measured Reeb graphs with solid and dashed edges.*

Note that the would-be compatibility condition should be consistent with that for the case of Morse functions constant on boundary components and considered in [5] as a limiting case. (Formally speaking, for simple Morse functions constants on the boundary have to be ruled

out and can be considered only in the limit, since the restriction of functions to the boundary is to be Morse.)

The goal here is to establish a classification theorem similar to Theorem 3.8: For a surface M , possibly with boundary, the mapping assigning the augmented measured Reeb graph Γ_F to a simple Morse function F should provide a one-to-one correspondence between simple Morse functions on M up to a symplectomorphism and augmented measured Reeb graphs compatible with M .

5.3 The boundary case: coadjoint orbits

To classify coadjoint orbits for the symplectomorphism group of a compact connected surface M with boundary one can employ a classification of Morse functions on such surfaces. Recall that regardless of the boundary, the regular dual $\mathfrak{svect}^*(M)$ of the Lie algebra $\mathfrak{svect}(M)$ of divergence-free vector fields on a surface M is identified with the space $\Omega^1(M)/d\Omega^0(M)$. Similarly to the no-boundary case, for a coset of 1-forms $\alpha \in \Omega^1(M)/d\Omega^0(M)$ consider the vorticity function $\mathbf{curl}[\alpha] := d\alpha/\omega$. Again confine ourselves to cosets of 1-forms $[\alpha] \in \mathfrak{svect}^*(M)$ of Morse type, i.e. to cosets with a simple Morse function $\mathbf{curl}[\alpha]$ on the surface with boundary.

Let $[\alpha] \in \mathfrak{svect}^*(M)$ be Morse-type, and let $F = \mathbf{curl}[\alpha]$. Now one can define a *circulation function* only on solid edges of the measured Reeb graph Γ_F . Indeed, let $\pi: M \rightarrow \Gamma_F$ be the natural projection. Take any point x lying in the interior of a solid edge $e \in \Gamma_F$. Then $\pi^{-1}(x)$ is an oriented circle ℓ_x . The integral of α over ℓ_x does not depend on the choice of a representative $\alpha \in [\alpha]$. (Note that preimages of points x in dashed edges e are segments, and integrals of α over them do depend on a representative $\alpha \in [\alpha]$, i.e. they are not well-defined objects on $\mathfrak{svect}^*(M)$.) Thus, we obtain a real-valued function \mathbf{c} defined on solid edges, which is an antiderivative of the density $f d\mu$ wherever it is defined and which satisfies the properties of Definition 3.9 at the vertices involving only solid edges. It is suggestive to define a circulation graph as an augmented measured Reeb graph equipped with a circulation function. It turns out however that the circulation function has to be further supplemented by additional information.

Example 5.8. Circulations, being integrals of 1-forms α , over boundary components are invariants of coadjoint action, i.e. Casimirs. (Since diffeomorphisms may interchange boundary components, the corresponding circulations over boundary components are to be considered up to permutation.) However, they do not enter the definition of circulation function on solid edges defined above. In particular, different surfaces corresponding to the same Reeb graph in Example 5.4, see Figure 5, have different number of boundary components, and hence different number of Casimirs from boundary conditions. On the other hand, since the Reeb graph consists of only dashed edges, one does not relate a circulation function to it, and hence one needs new options for “storing” the information about the coadjoint orbits.

Problem 5.9. *How to supplement the circulation function on solid edges by extending it to dashed ones to fully describe coadjoint orbits by means of “circulation augmented measured Reeb graphs”?*

One possibility would be to define circulation functions as integrals of cosets over level segments (rather than circle levels) by closing up the segments using boundary arcs. We expect that the latter approach works at least in the case of one boundary component.

Remark 5.10. Note that a partial list of Casimirs in the 2D case with boundary can be described exactly in the same way as in the no-boundary case: one has to consider all moments for the measure on each edge, either solid or dashed, of the measured Reeb graph for the vorticity function. The discrete information on the Reeb graph does not affect the definition of families of continuous Casimirs. However, the list of additional circulations yet needs to be detailed.

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